

# EXTERIOR NAVIER-STOKES FLOWS FOR BOUNDED DATA

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**ABSTRACT.** We prove unique existence of mild solutions on  $L_\sigma^\infty$  for the Navier-Stokes equations in an exterior domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , subject to the non-slip boundary condition.

## 1. INTRODUCTION

We consider the initial-boundary value problem of the Navier-Stokes equations in an exterior domain  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ :

$$(1.1) \quad \begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} u &= 0 & \text{in } \Omega \times (0, T), \\ u &= 0 & \text{on } \partial\Omega \times (0, T), \\ u &= u_0 & \text{on } \Omega \times \{t = 0\}. \end{aligned}$$

There is a large literature on the solvability of the exterior problem for initial data decaying at space infinity. However, a few results are available for non-decaying data. A typical example of non-decaying flow is a stationary solution of (1.1) having a finite Dirichlet integral, called *D*-solution [24]. It is known that *D*-solutions are bounded in  $\Omega$  and asymptotically constant as  $|x| \rightarrow \infty$ ; see Remarks 1.2 (ii). In this paper, we do not impose on  $u_0$  conditions at space infinity.

The purpose of this paper is to establish a solvability of (1.1) for merely bounded initial data. We set the solenoidal  $L^\infty$ -space,

$$L_\sigma^\infty(\Omega) = \left\{ f \in L^\infty(\Omega) \mid \int_\Omega f \cdot \nabla \varphi dx = 0 \text{ for } \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

by the homogeneous Sobolev space  $\hat{W}^{1,1}(\Omega) = \{\varphi \in L_{\text{loc}}^1(\Omega) \mid \nabla \varphi \in L^1(\Omega)\}$ . For exterior domains, the space  $L_\sigma^\infty$  agrees with the space of all bounded divergence-free vector fields, whose normal trace is vanishing on  $\partial\Omega$  [4]. The  $L^\infty$ -type solvability for (1.1) is recently established on  $C_{0,\sigma}$  in the previous work of the author [1], where  $C_{0,\sigma}$  is the  $L^\infty$ -closure of  $C_{c,\sigma}^\infty$ , the space of all smooth solenoidal vector fields with compact support in  $\Omega$ . Since the condition  $u_0 \in C_{0,\sigma}$  imposes the decay  $u_0 \rightarrow 0$  as  $|x| \rightarrow \infty$ , we develop an existence theorem for non-decaying space  $L_\sigma^\infty$ , which in particular includes asymptotically constant vector fields. Moreover, the space  $L_\sigma^\infty$  includes vector fields rotating at space infinity; see Remarks 1.2 (iv). When  $\Omega$  is the whole space [16] or a half space [33], [7], the existence of mild solutions of (1.1) on  $L_\sigma^\infty$  is proved by explicit formulas of the Stokes semigroup. In

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*Date:* June 17, 2016.

*2010 Mathematics Subject Classification.* 35Q35, 35K90.

*Key words and phrases.* Navier-Stokes equations, bounded function spaces, exterior problem, *D*-solutions.

this paper, we prove unique existence of mild solutions on  $L^\infty_\sigma$  for exterior domains based on  $L^\infty$ -estimates of the Stokes semigroup [4], [2].

To state a result, let  $S(t)$  denote the Stokes semigroup. It is proved in [4] that  $S(t)$  is an analytic semigroup on  $L^\infty_\sigma$  for exterior domains of class  $C^3$ . Let  $\mathbb{P}$  denote the Helmholtz projection. We write  $\operatorname{div} F = (\sum_{i=1}^n \partial_i F_{ij})$  for matrix-valued functions  $F = (F_{ij})$ . It is proved in [2] that the composition operator  $S(t)\mathbb{P}\operatorname{div}$  satisfies an estimate of the form

$$(1.2) \quad \|S(t)\mathbb{P}\operatorname{div} F\|_{L^\infty(\Omega)} \leq \frac{C_\alpha}{t^{\frac{1-\alpha}{2}}} \|F\|_{L^\infty(\Omega)}^{1-\alpha} \|\nabla F\|_{L^\infty(\Omega)}^\alpha,$$

for  $F \in C_0^1 \cap W^{1,2}(\Omega)$ ,  $t \leq T_0$  and  $\alpha \in (0, 1)$ . Here,  $W^{1,2}(\Omega)$  denotes the Sobolev space and  $C_0^1(\Omega)$  denotes the  $W^{1,\infty}$ -closure of  $C_c^\infty(\Omega)$ , the space of all smooth functions with compact support in  $\Omega$ . Although the projection  $\mathbb{P}$  may not act as a bounded operator on  $L^\infty$ , the  $L^\infty$ -estimate (1.2) implies that the composition  $S(t)\mathbb{P}\operatorname{div}$  is uniquely extendable to a bounded operator from  $C_0^1$  to  $C_{0,\sigma}$ . Note that  $F \in C_0^1$  imposes a decay condition at space infinity. Thus the extension to  $C_0^1$  is not sufficient for studying non-decaying solutions. In this paper, we prove that the composition  $S(t)\mathbb{P}\operatorname{div}$  is uniquely extendable to a bounded operator  $\overline{S(t)\mathbb{P}\operatorname{div}}$  from the non-decaying space  $W_0^{1,\infty}$  to  $L^\infty_\sigma$ , where  $W_0^{1,\infty}$  is the space of all functions in  $W^{1,\infty}$  vanishing on  $\partial\Omega$ .

By means of the new extension, we study the integral equation on  $L^\infty_\sigma$  of the form

$$(1.3) \quad u(t) = S(t)u_0 - \int_0^t \overline{S(t-s)\mathbb{P}\operatorname{div}}(uu)(s)ds.$$

Here,  $uu = (u_i u_j)$  is the tensor product. We call solutions of (1.3) mild solution on  $L^\infty_\sigma$ . Since the projection  $\mathbb{P}$  may not be bounded on  $L^\infty$ , the extension  $\overline{S(t)\mathbb{P}\operatorname{div}}$  is not expressed by the individual operators. We thus prove that mild solutions satisfy (1.1) by using a weak form. Let  $C_{c,\sigma}^\infty(\Omega \times [0, T])$  denote the space of all smooth solenoidal vector fields with compact support in  $\Omega \times [0, T]$ . Let  $C([0, T]; X)$  (resp.  $C_w([0, T]; X)$ ) denote the space of all (resp. weakly-star) continuous functions from  $[0, T]$  to a Banach space  $X$ . Let  $BUC_\sigma(\Omega)$  denote the space of all solenoidal vector fields in  $BUC(\Omega)$  vanishing on  $\partial\Omega$ , where  $BUC(\Omega)$  is the space of all bounded uniformly continuous functions in  $\overline{\Omega}$ . Let  $[\cdot]_\Omega^{(\beta)}$  denote the  $\beta$ -th Hölder semi-norm in  $\overline{\Omega}$ . The main result of this paper is the following:

**Theorem 1.1.** *Let  $\Omega$  be an exterior domain with  $C^3$ -boundary in  $\mathbb{R}^n$ ,  $n \geq 2$ . For  $u_0 \in L^\infty_\sigma$ , there exist  $T \geq \varepsilon/\|u_0\|_\infty^2$  and a unique mild solution  $u \in C_w([0, T]; L^\infty)$  such that*

$$(1.4) \quad \int_0^T \int_\Omega (u \cdot (\partial_t \varphi + \Delta \varphi) + uu : \nabla \varphi) dx dt = - \int_\Omega u_0(x) \cdot \varphi(x, 0) dx$$

for all  $\varphi \in C_{c,\sigma}^\infty(\Omega \times [0, T])$ , with some constant  $\varepsilon = \varepsilon_\Omega$ . The solution  $u$  satisfies

$$(1.5) \quad \sup_{0 < t \leq T} \left\{ \|u\|_{L^\infty(\Omega)}(t) + t^{\frac{1}{2}} \|\nabla u\|_{L^\infty(\Omega)}(t) + t^{\frac{1+\beta}{2}} [\nabla u]_\Omega^{(\beta)}(t) \right\} \leq C_1 \|u_0\|_{L^\infty(\Omega)},$$

$$(1.6) \quad \sup_{x \in \Omega} \left\{ [u]_{[\delta, T]}^{(\gamma)}(x) + [\nabla u]_{[\delta, T]}^{(\frac{\gamma}{2})}(x) \right\} \leq C_2 \|u_0\|_{L^\infty(\Omega)},$$

for  $\beta, \gamma \in (0, 1)$  and  $\delta \in (0, T)$  with the constant  $C_1$ , independent of  $u_0$  and  $T$ . The constant  $C_2$  depends on  $\gamma, \delta$  and  $T$ . If  $u_0 \in BUC_\sigma$ ,  $u, t^{1/2}\nabla u \in C([0, T]; BUC)$  and  $t^{1/2}\nabla u$  vanishes at time zero.

**Remarks 1.2.** (i) (Blow-up rate) By the estimate of the existence time in Theorem 1.1, we obtain a blow-up rate of mild solutions  $u \in C_w([0, T_*); L^\infty)$  of the form

$$\|u\|_{L^\infty(\Omega)} \geq \frac{\varepsilon'}{\sqrt{T_* - t}} \quad \text{for } t < T_*,$$

with  $\varepsilon' = \varepsilon^{1/2}$ , where  $t = T_*$  is the blow-up time. The above blow-up estimate was first proved by Leray [25] for  $\Omega = \mathbb{R}^3$ . See [16] for  $n \geq 3$  and [33] ([27], [7]) for a half space. The statement of Theorem 1.1 is valid also for a half space and improves regularity properties of mild solutions on  $L^\infty_\sigma$  proved in [33], [7].

(ii) ( $D$ -solutions) In [24], Leray proved the existence of  $D$ -solutions  $u$  satisfying  $u - u_\infty \in L^6(\Omega)$  for  $u_\infty \in \mathbb{R}^3$  in the exterior domain  $\Omega \subset \mathbb{R}^3$ . His construction is based on an approximation for  $R \rightarrow \infty$  of the problem

$$\begin{aligned} -\Delta u_R + u_R \cdot \nabla u_R + \nabla p_R &= 0 & \text{in } \Omega_R, \\ \operatorname{div} u_R &= 0 & \text{in } \Omega_R, \\ u_R &= 0 & \text{on } \partial\Omega, \\ u_R &= u_\infty & \text{on } \{|x| = R\}, \end{aligned}$$

for  $\Omega_R = \Omega \cap \{|x| < R\}$  ([23, Chapter 5, Theorem 5]). See also [13, Theorem 3.2] ([14, Theorem X.4.1]) for a different construction. If the Dirichlet integral is finite, stationary solutions of (1.1) are locally bounded in  $\overline{\Omega}$  (e.g., [14, Theorem X.1.1]). Moreover,  $D$ -solutions are bounded as  $|x| \rightarrow \infty$  by  $u - u_\infty \in L^6(\Omega)$ . Thus,  $D$ -solutions are elements of  $L^\infty_\sigma$  for  $n = 3$ .

When  $n = 2$ , more analysis is needed for information about the behavior as  $|x| \rightarrow \infty$  since a finite Dirichlet integral does not imply decays at space infinity (e.g.,  $u = (\log |x|)^\alpha$  for  $0 < \alpha < 1/2$ ). Leray's construction gives  $D$ -solutions also in  $\Omega \subset \mathbb{R}^2$ . It is proved in [18] ([19]) that Leray's solutions are bounded in  $\overline{\Omega}$  and converge to some constant  $\bar{u}_\infty$  in the sense that  $\int_0^{2\pi} |u(re_r) - \bar{u}_\infty| d\theta \rightarrow 0$  as  $r \rightarrow \infty$ , where  $(r, \theta)$  is the polar coordinate and  $e_r = (\cos \theta, \sin \theta)$ . Moreover, every  $D$ -solutions are bounded and asymptotically constant in the above sense [6, Theorem 12]. Thus,  $D$ -solutions are elements of  $L^\infty_\sigma$  also for  $n = 2$ . Theorem 1.1 yields a local solvability of (1.1) around  $D$ -solutions without imposing decay conditions for initial disturbance.

(iii) (Global well-posedness for  $n = 2$ ) It is well known that the exterior problem (1.1) for  $n = 2$  is globally well-posed for initial data having finite energy, e.g., [22]. However, global well-posedness is unknown for non-decaying data  $u_0 \in L^\infty_\sigma$ . For the whole space, the vorticity  $\omega = \partial_1 u^2 - \partial_2 u^1$  satisfies the a priori estimate

$$\|\omega\|_{L^\infty(\mathbb{R}^2)} \leq \|\omega_0\|_{L^\infty(\mathbb{R}^2)} \quad t > 0.$$

It is proved in [17] that the Cauchy problem of (1.1) for  $n = 2$  is globally well-posed for  $u_0 \in L^\infty_\sigma$  based on the local solvability result in [16]. We proved a local solvability on  $L^\infty_\sigma$  for exterior domains. Note that global solutions exist for rotationally symmetric initial data  $u_0 \in L^\infty_\sigma$ ; see below (iv).

(iv) (Rotating flows) An example of  $u_0 \in L^\infty_\sigma$  which is not asymptotically constant is a vector field rotating at space infinity. For example, we consider the two-dimensional unit

disk  $\Omega^c$  centered at the origin and a rotationally symmetric initial data  $u_0 = u_0^\theta(r)e_\theta(\theta)$  for  $e_\theta(\theta) = (-\sin \theta, \cos \theta)$ . Observe that  $u_0$  is a solenoidal vector field in  $\Omega$  and a direction of  $u_0$  varies for  $\theta \in [0, 2\pi]$  and  $u_0^\theta \in L^\infty(1, \infty)$ . Solutions of (1.1) for  $u_0$  are rotationally symmetric and given by

$$u = e^{t\Delta_D} u_0 \quad p = \int_1^{|x|} \frac{|u|^2}{r} dr,$$

where  $\Delta_D$  denotes the Laplace operator subject to the Dirichlet boundary condition. The solution  $u$  is bounded in  $\Omega \times (0, \infty)$  and non-decaying as  $|x| \rightarrow \infty$ .

(v) (Associated pressure) We invoke that the associated pressure of mild solutions on  $L^p$  ( $p \geq n$ ) is determined by the projection operator  $\mathbb{Q} = I - \mathbb{P}$  and

$$\nabla p = \mathbb{Q}\Delta u - \mathbb{Q}(u \cdot \nabla u).$$

Since the projection  $\mathbb{Q}$  may not be bounded on  $L^\infty$ , this representation is no longer available for mild solutions on  $L^\infty_\sigma$ . When  $\Omega = \mathbb{R}^n$  or  $\mathbb{R}^n_+$ , the projection  $\mathbb{Q}$  has explicit kernels and we are able to find associated pressure of mild solutions on  $L^\infty$ ; see [16] for  $\Omega = \mathbb{R}^n$  and [33], [27], [7] for  $\Omega = \mathbb{R}^n_+$ . Although explicit kernels are not available for exterior domains, we are able to find the associated pressure of mild solutions on  $L^\infty$ . We set

$$(1.7) \quad \nabla p = \mathbb{K}W - \mathbb{Q}\operatorname{div} F$$

for  $W = -(\nabla u - \nabla^T u)n_\Omega$  and  $F = uu$ , where  $n_\Omega$  is the unit outward normal on  $\partial\Omega$  and  $\mathbb{K}$  is a solution operator of the homogeneous Neumann problem (*harmonic-pressure operator*) [4, Remarks 4.3 (ii)]. Note that  $W = -\operatorname{curl} u \times n_\Omega$  for  $n = 3$ . The operators  $\mathbb{K}$  and  $\mathbb{Q}\operatorname{div}$  act for bounded functions and the associated pressure on  $L^\infty$  is uniquely determined by (1.7) in the sense of distribution; see Remark 3.5 for a detailed discussion.

For asymptotically constant initial data  $u_0$  (i.e.,  $u_0 \rightarrow u_\infty$  as  $|x| \rightarrow \infty$ ), local solvability of (1.1) for  $n = 3$  is proved in [29, Theorem 5.2] by means of the Oseen semigroup. In the paper, the problem (1.1) is reduced to an initial-boundary problem for decaying data by shifting  $u$  by a constant  $u_\infty$ . Our analysis is based on the  $L^\infty$ -estimates of the Stokes semigroup which yields a local-in-time solvability of (1.1) without conditions for  $u_0$  at space infinity.

The  $L^\infty$ -theory for the Cauchy problem of the Navier-Stokes equations is developed by Knightly [20], [21], Cannon and Knightly [8], Cannone [9] ([10]) and Giga et al. [16]. For the whole space, mild solutions on  $L^\infty$  are smooth and satisfy (1.1) in a classical sense [16]. For a half space, mild solutions on  $L^\infty$  are constructed in [33] (see also [27], [7]). There are a few results on solvability of the exterior problem for non-decaying data. In [15], unique existence of continuous solutions of (1.1) for  $n \geq 3$  is proved for non-decaying and Hölder continuous initial data. The result is extended in [28] for merely bounded  $u_0 \in L^\infty_\sigma$  and  $n \geq 3$  by using  $L^\infty$ -estimates of the Stokes semigroup [3], [4]. Note that mild solutions on  $L^\infty_\sigma$  are not constructed without the composition operator  $\bar{S}(t)\mathbb{P}\operatorname{div}$ . We proved the unique existence of mild solutions on  $L^\infty_\sigma$ , which in particular yields a local-in-time solvability for  $n = 2$ . The integral form (1.3) is fundamental for studying solutions of (1.1). We expect that mild solutions on  $L^\infty$  are sufficiently smooth and satisfy (1.1) in a classical sense.

The article is organized as follows. In Section 2, we extend the composition operator  $S(t)\mathbb{P}\text{div}$  to a bounded operator from  $W_0^{1,\infty}$  to  $L_\sigma^\infty$  by approximation as we did the Stokes semigroup in [4]. We extend  $S(t)\mathbb{P}\text{div}$  as a solution operator  $F \mapsto v(\cdot, t)$  for solutions  $(v, q)$  of the Stokes equations for  $v_0 = \mathbb{P}\text{div } F$ . Note that  $v_0 = \mathbb{P}\text{div } F$  for  $F \in W_0^{1,\infty}$  is not an element of  $L^\infty$  in general since the projection  $\mathbb{P}$  is not bounded on  $L^\infty$ . We understand  $v_0 = \mathbb{P}\text{div } F$  as distribution by using the fact that  $\nabla \mathbb{P}\varphi \in L^1$  for  $\varphi \in C_c^\infty$  (Lemma A.1). We approximate  $F \in W_0^{1,\infty}$  by a sequence  $\{F_m\} \subset C_c^\infty$  locally uniformly in  $\overline{\Omega}$  and obtain a unique extension  $\overline{S(t)\mathbb{P}\text{div}} : F \mapsto v(\cdot, t)$  by a limit  $v$  of the sequence  $v_m = S(t)\mathbb{P}\text{div } F_m$ .

In Section 3, we prove Theorem 1.1. We approximate initial data  $u_0 \in L_\sigma^\infty$  by a sequence  $\{u_{0,m}\} \subset C_{c,\sigma}^\infty$  satisfying  $u_{0,m} \rightarrow u_0$  a.e. in  $\Omega$  and  $\|u_{0,m}\|_\infty \leq C\|u_0\|_\infty$ . Since the property of mild solutions (1.4) may not follow from a direct iteration argument on  $L_\sigma^\infty$ , we construct mild solutions by approximation. We apply an existence theorem on  $C_{0,\sigma}$  [1] and construct a sequence of mild solutions  $u_m \in C([0, T]; C_{0,\sigma})$  satisfying (1.4)-(1.6) for  $u_{0,m} \in C_{0,\sigma}$ . We prove that  $u_m$  subsequently converges to a mild solution  $u$  for  $u_0 \in L_\sigma^\infty$  locally uniformly in  $\overline{\Omega} \times (0, T]$ .

In Appendix A, we show that  $\nabla \mathbb{P}\varphi \in L^1$  for  $\varphi \in C_c^\infty$  by means of the layer potential.

## 2. AN EXTENSION OF THE COMPOSITION OPERATOR

In this section, we prove that the composition operator  $S(t)\mathbb{P}\partial$  is uniquely extendable to a bounded operator from  $W_0^{1,\infty}$  to  $L_\sigma^\infty$ . We prove unique existence of solutions of the Stokes equations for initial data  $v_0 = \mathbb{P}\partial f$ ,  $f \in W_0^{1,\infty}$ , and extend the composition as a solution operator  $S(t)\mathbb{P}\partial : f \mapsto v(\cdot, t)$ . In what follows,  $\partial = \partial_j$  indiscriminately denotes the spatial derivatives for  $j = 1, \dots, n$ .

**2.1. The Stokes system.** We consider the Stokes equations,

$$\begin{aligned} \partial_t v - \Delta v + \nabla q &= 0 & \text{in } \Omega \times (0, T), \\ \text{div } v &= 0 & \text{in } \Omega \times (0, T), \\ v &= 0 & \text{on } \partial\Omega \times (0, T), \\ v &= v_0 & \text{on } \Omega \times \{t = 0\}. \end{aligned} \tag{2.1}$$

We set the norm

$$N(v, q)(x, t) = |v(x, t)| + t^{\frac{1}{2}} |\nabla v(x, t)| + t |\nabla^2 v(x, t)| + t |\partial_t v(x, t)| + t |\nabla q(x, t)|.$$

Let  $d(x)$  denote the distance from  $x \in \Omega$  to  $\partial\Omega$ . Let  $(v, \nabla q) \in C^{2+\mu, 1+\frac{\mu}{2}}(\overline{\Omega} \times (0, T]) \times C^{\mu, \frac{\mu}{2}}(\overline{\Omega} \times (0, T])$ ,  $\mu \in (0, 1)$ , satisfy the equations and the boundary condition of (2.1). We say that  $(v, q)$  is a solution of (2.1) for  $v_0 = \mathbb{P}\partial f$ ,  $f \in W_0^{1,\infty}(\Omega)$ , if

$$\sup_{0 < t \leq T} \left\{ t^\gamma \|N(v, q)\|_\infty(t) + t^{\gamma+\frac{1}{2}} \|d\nabla q\|_\infty(t) \right\} < \infty, \tag{2.2}$$

for some  $\gamma \in [0, 1/2)$  and

$$\int_0^T \int_\Omega (v \cdot (\partial_t \varphi + \Delta \varphi) - \nabla q \cdot \varphi) dx dt = \int_\Omega f \cdot \partial \mathbb{P}\varphi_0 dx \tag{2.3}$$

for all  $\varphi \in C_c^\infty(\Omega \times [0, T])$  with  $\varphi_0(x) = \varphi(x, 0)$ . The left-hand side is finite since  $\varphi(\cdot, t)$  is supported in  $\Omega$  and  $\gamma < 1/2$ . The right-hand side is finite since  $\partial \mathbb{P}\varphi_0$  is integrable in  $\Omega$  for  $\varphi_0 \in C_c^\infty(\Omega)$  by Lemma A.1. As explained later in Remarks 2.9 (i), the operator  $\mathbb{P}\partial$  is uniquely extendable for  $f \in W_0^{1,\infty}$  and we are able to define  $\mathbb{P}\partial f$  in the sense of distribution. The goal of this section is to prove:

**Theorem 2.1.** *Let  $\Omega$  be an exterior domain with  $C^3$ -boundary. Let  $T > 0$ . For  $v_0 = \mathbb{P}\partial f$ ,  $f \in W_0^{1,\infty}(\Omega)$ , there exists a unique solution  $(v, q)$  of (2.1) satisfying*

$$(2.4) \quad \sup_{0 < t \leq T} \left\{ t^\gamma \|N(v, q)\|_\infty(t) + t^{\gamma+\frac{1}{2}} \|d\nabla q\|_\infty(t) \right\} \leq C \|f\|_\infty^{1-\alpha} \|f\|_{1,\infty}^\alpha,$$

for  $\alpha \in (0, 1)$  with  $\gamma = (1 - \alpha)/2$  and some constant  $C$ , depending on  $\alpha, T$  and  $\Omega$ .

Theorem 2.1 implies the following:

**Theorem 2.2.** *The composition operator  $S(t)\mathbb{P}\partial$  is uniquely extendable to a bounded operator  $\bar{S}(t)\mathbb{P}\partial$  from  $W_0^{1,\infty}(\Omega)$  to  $L^\infty(\Omega)$  together with the estimate*

$$(2.5) \quad \sup_{0 < t \leq T} t^{\gamma+\frac{|k|}{2}+s} \left\| \partial_t^s \partial_x^k \bar{S}(t)\mathbb{P}\partial f \right\|_\infty \leq C \|f\|_\infty^{1-\alpha} \|f\|_{1,\infty}^\alpha,$$

for  $f \in W_0^{1,\infty}(\Omega)$ ,  $0 \leq 2s + |k| \leq 2$  and  $\alpha \in (0, 1)$  with  $\gamma = (1 - \alpha)/2$ .

**2.2. Hölder estimates and uniqueness.** In order to prove Theorem 2.1, we recall local Hölder estimates and a uniqueness result for the Stokes equations. In the subsequent section, we give a proof for Theorem 2.1 by approximation.

We set the Hölder semi-norm

$$[f]_Q^{(\mu, \frac{\mu}{2})} = \sup_{t \in (0, T]} [f]_\Omega^{(\mu)}(t) + \sup_{x \in \Omega} [f]_{(0, T]}^{(\frac{\mu}{2})}(x), \quad \mu \in (0, 1),$$

for  $Q = \Omega \times (0, T]$ . We set

$$N = \sup_{\delta \leq t \leq T} \|N(v, q)\|_{L^\infty(\Omega)}(t)$$

for solutions  $(v, q)$  of (2.1). The following local Hölder estimate is proved in [3, Proposition 3.2 and Theorem 3.4] based on the Schauder estimates for the Stokes equations [35] ([32], [34]).

**Proposition 2.3.** *Let  $\Omega$  be an exterior domain with  $C^3$ -boundary.*

(i) *(Interior estimates) For  $\mu \in (0, 1)$ ,  $\delta > 0$ ,  $T > 0$ ,  $R > 0$ , there exists a constant  $C = C(\mu, \delta, T, R, d)$  such that*

$$(2.6) \quad [\nabla^2 v]_{Q'}^{(\mu, \frac{\mu}{2})} + [v_t]_{Q'}^{(\mu, \frac{\mu}{2})} + [\nabla q]_{Q'}^{(\mu, \frac{\mu}{2})} \leq CN$$

holds for all solutions  $(v, q)$  of (2.1) for  $Q' = B_{x_0}(R) \times (2\delta, T]$  and  $x_0 \in \Omega$  satisfying  $\overline{B_{x_0}(R)} \subset \Omega$ , where  $d$  denotes the distance from  $B_{x_0}(R)$  to  $\partial\Omega$ .

(ii) *(Estimates up to the boundary) There exists  $R_0 > 0$  such that for  $\mu \in (0, 1)$ ,  $\delta > 0$ ,  $T > 0$  and  $R \leq R_0$ , there exists a constant  $C$  depending on  $\mu, \delta, T, R$  and  $C^3$ -regularity*

of  $\partial\Omega$  such that (2.6) holds for all solutions  $(v, q)$  of (2.1) for  $Q' = \Omega_{x_0, R} \times (2\delta, T]$  and  $\Omega_{x_0, R} = B_{x_0}(R) \cap \Omega$ ,  $x_0 \in \partial\Omega$ .

We observe the uniqueness of solutions for (2.1). The uniqueness of the Stokes equations (2.1) for  $v_0 \in L^\infty_\sigma$  in an exterior domain is proved based on the uniqueness result in a half space [33] by a blow-up argument; see [4, Lemma 2.12]. In order to prove Theorem 2.1, we need a stronger uniqueness result since solutions of (2.1) for  $v_0 = \mathbb{P}\partial f$ ,  $f \in W_0^{1,\infty}$ , may not be bounded at  $t = 0$ . The corresponding uniqueness result for a half space is recently proved in [2, Theorem 5.1]. We deduce the result for exterior domains by the same blow-up argument as we did in [4].

**Proposition 2.4.** *Let  $\Omega$  be an exterior domain with  $C^3$ -boundary. Let  $(v, \nabla q) \in C^{2,1}(\overline{\Omega} \times (0, T]) \times C(\overline{\Omega} \times (0, T])$  satisfy the equations and the boundary condition of (2.1), and (2.2) for some  $\gamma \in [0, 1/2)$ . Assume that*

$$\int_0^T \int_\Omega (v \cdot (\partial_t \varphi + \Delta \varphi) - \nabla q \cdot \varphi) dx dt = 0,$$

for all  $\varphi \in C_c^\infty(\Omega \times [0, T])$ . Then,  $v \equiv 0$  and  $\nabla q \equiv 0$ .

**2.3. Approximation.** We prove Theorem 2.1. We show existence of solutions for the Stokes equations (2.1) for  $v_0 = \mathbb{P}\partial f$ ,  $f \in W_0^{1,\infty}$ , by approximation. We approximate  $f \in W_0^{1,\infty}$  by elements of  $C_c^\infty$  locally uniformly in  $\overline{\Omega}$ .

**Lemma 2.5.** *Let  $\Omega$  be an exterior domain with Lipschitz boundary. There exist constants  $C_1, C_2$  such that for  $f \in W_0^{1,\infty}(\Omega)$  there exists a sequence of functions  $\{f_m\}_{m=1}^\infty \subset C_c^\infty(\Omega)$  such that*

$$(2.7) \quad \begin{aligned} \|f_m\|_\infty &\leq C_1 \|f\|_\infty, \\ \|\nabla f_m\|_\infty &\leq C_2 \|f\|_{1,\infty}, \\ f_m &\rightarrow f \quad \text{locally uniformly in } \overline{\Omega} \quad \text{as } m \rightarrow \infty. \end{aligned}$$

The proof of Lemma 2.5 is reduced to the whole space and bounded domains.

**Proposition 2.6.** *The statement of Lemma 2.5 holds when  $\Omega = \mathbb{R}^n$  with  $C_1 = 1$ .*

*Proof.* We cutoff the function  $f \in W^{1,\infty}(\mathbb{R}^n)$ . Let  $\theta \in C_c^\infty[0, \infty)$  be a cut-off function satisfying  $\theta \equiv 1$  in  $[0, 1]$ ,  $\theta \equiv 0$  in  $[2, \infty)$  and  $0 \leq \theta \leq 1$ . We set  $\theta_m(x) = \theta(|x|/m)$  for  $m \geq 1$  so that  $\theta_m \equiv 1$  for  $|x| \leq m$  and  $\theta_m \equiv 0$  for  $|x| \geq 2m$ . Then,  $f_m = f\theta_m$  satisfies (2.7).  $\square$

**Proposition 2.7.** *Let  $\Omega$  be a bounded domain with Lipschitz boundary. There exists a constant  $C_3$  such that for  $f \in W_0^{1,\infty}(\Omega)$  there exists a sequence of functions  $\{f_m\}_{m=1}^\infty \subset C_c^\infty(\Omega)$  such that*

$$(2.8) \quad \begin{aligned} \|\nabla f_m\|_\infty &\leq C_3 \|\nabla f\|_\infty \\ f_m &\rightarrow f \quad \text{uniformly in } \overline{\Omega} \quad \text{as } m \rightarrow \infty. \end{aligned}$$

*Proof.* We begin with the case when  $\Omega$  is star-shaped, i.e.,  $\lambda\Omega_{x_0} \subset \overline{\Omega}$  for some  $x_0 \in \Omega$  and all  $\lambda < 1$ , where  $\lambda\Omega_{x_0} = \{x_0 + \lambda(x - x_0) \mid x \in \Omega\}$ . We may assume  $x_0 = 0 \in \Omega$  and  $\lambda\Omega \subset \overline{\Omega}$  by translation.

For  $f \in W_0^{1,\infty}(\Omega)$ , we set

$$f_\lambda(x) = \begin{cases} f(x/\lambda) & x \in \lambda\Omega, \\ 0 & x \in \Omega \setminus \overline{\lambda\Omega}. \end{cases}$$

Then,  $f_\lambda$  is in  $W^{1,\infty}(\Omega)$  since  $f$  is vanishing on  $\partial\Omega$ . It follows that

$$\|\nabla f_\lambda\|_\infty \leq \frac{1}{\lambda} \|\nabla f\|_\infty,$$

and  $f_\lambda \rightarrow f$  uniformly in  $\overline{\Omega}$  as  $\lambda \rightarrow 1$ . By a mollification of  $f_\lambda$ , we obtain a sequence  $\{f_m\} \subset C_c^\infty(\Omega)$  satisfying (2.8) with  $C_3 = 2$ .

For general  $\Omega$ , we take an open covering  $\{D_j\}_{j=1}^N$  so that  $\overline{\Omega} \subset \cup_{j=1}^N D_j$  and  $\Omega_j = \Omega \cap D_j$  is Lipschitz and star-shaped for some  $x_j \in \Omega_j$  [14, Lemma II 1.3]. We take a partition of unity  $\{\xi_j\}_{j=1}^N \subset C_c^\infty(\mathbb{R}^n)$  such that  $\sum_{j=1}^N \xi_j = 1$ ,  $0 \leq \xi_j \leq 1$ ,  $\text{spt } \xi_j \subset \overline{D_j}$  and set

$$f = \sum_{j=1}^N f_j, \quad f_j = f \xi_j.$$

Since  $\text{spt } f_j \subset \overline{\Omega_j}$ ,  $\xi_j = 0$  on  $\partial D_j$  and  $f = 0$  on  $\partial\Omega$ ,  $f_j$  is in  $W_0^{1,\infty}(\Omega_j)$ . Since  $\Omega_j$  is star-shaped for some  $x_j \in \Omega_j$ , there exists  $\{f_{j,m}\} \subset C_c^\infty(\Omega_j)$  satisfying (2.8) in  $\Omega_j$  with  $C_3 = 2$ . We extend  $f_{j,m} \in C_c^\infty(\Omega_j)$  to  $\Omega \setminus \overline{\Omega_j}$  by the zero extension (still denoted by  $f_{j,m}$ ) and set  $f_m = \sum_{j=1}^N f_{j,m}$ . Then,  $f_m \in C_c^\infty(\Omega)$  converges to  $f$  uniformly in  $\overline{\Omega}$ . We estimate

$$\|\nabla f_m\|_{L^\infty(\Omega)} \leq \sum_{j=1}^N \|\nabla f_{j,m}\|_{L^\infty(\Omega_j)} \leq 2 \sum_{j=1}^N \|\nabla f_j\|_{L^\infty(\Omega_j)}.$$

Since  $\nabla f_j = \nabla f \xi_j + f \nabla \xi_j$  and

$$\|f\|_{L^\infty(\Omega)} \leq C_P \|\nabla f\|_{L^\infty(\Omega)},$$

by the Poincaré inequality (e.g., [11, 5.8.1 Theorem 1]), we obtain

$$\|\nabla f_m\|_{L^\infty(\Omega)} \leq C \|\nabla f\|_{L^\infty(\Omega)}.$$

Thus,  $\{f_m\} \subset C_c^\infty(\Omega)$  satisfies (2.8). The proof is complete.  $\square$

*Proof of Lemma 2.5.* The assertion follows from Propositions 2.6 and 2.7.  $\square$

We recall the a priori estimate of  $S(t)\mathbb{P}\partial$  for  $f \in C_c^\infty(\Omega)$  [2, Theorem 1.2].

**Proposition 2.8.** *There exists a constant  $C$  such that*

$$(2.9) \quad \sup_{0 \leq t \leq T} t^{\gamma + \frac{|k|}{2} + s} \left\| \partial_t^s \partial_x^k S(t) \mathbb{P} \partial f \right\|_\infty \leq C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha$$

for  $f \in C_c^\infty(\Omega)$ ,  $0 \leq 2s + |k| \leq 2$  and  $\alpha \in (0, 1)$  with  $\gamma = (1 - \alpha)/2$ .



*Proof of Theorem 2.1.* For  $f \in W_0^{1,\infty}$ , we take a sequence  $\{f_m\} \subset C_c^\infty$  satisfying (2.7). For  $v_{0,m} = \mathbb{P}\partial f_m$ , there exists a solution of the Stokes equations  $(v_m, q_m)$  satisfying

$$\int_0^T \int_\Omega (v_m \cdot (\partial_t \varphi + \Delta \varphi) - \nabla q_m \cdot \varphi) dx dt = \int_\Omega f_m \cdot \partial \mathbb{P} \varphi_0 dx,$$

for  $\varphi \in C_c^\infty(\Omega \times [0, T])$ . By (2.9) and (2.7), there exists a constant  $C$  independent of  $m \geq 1$  such that

$$\sup_{0 \leq t \leq T} \left\{ t^\gamma \|N(v_m, q_m)\|_\infty(t) + t^{\gamma+\frac{1}{2}} \|d\nabla q_m\|_\infty(t) \right\} \leq C \|f\|_\infty^{1-\alpha} \|f\|_{1,\infty}^\alpha.$$

We apply Proposition 2.3 and observe that there exists a subsequence of  $(v_m, q_m)$  such that  $(v_m, q_m)$  converges to a limit  $(v, q)$  locally uniformly in  $\overline{\Omega} \times (0, T]$  together with  $\nabla v_m$ ,  $\nabla^2 v_m$ ,  $\partial_t v_m$  and  $\nabla q_m$ . By sending  $m \rightarrow \infty$ , we obtain a solution  $(v, q)$  of (2.1) for  $v_0 = \mathbb{P}\partial f$ . By Proposition 2.4, the limit  $(v, q)$  is unique. We proved the unique existence of solutions of (2.1) for  $v_0 = \mathbb{P}\partial f$  and  $f \in W_0^{1,\infty}$  satisfying (2.4). The proof is now complete.  $\square$

**Remarks 2.9.** (i) By the approximation (2.7) we are able to extend the operator  $\mathbb{P}\partial$  for  $f \in W_0^{1,\infty}$ . We take a sequence  $\{f_m\} \subset C_c^\infty$  satisfying (2.7) by Lemma 2.5 and observe that  $v_{0,m} = \mathbb{P}\partial f_m$  satisfies

$$(v_{0,m}, \varphi) = -(f_m, \partial \mathbb{P} \varphi) \quad \text{for } \varphi \in C_c^\infty(\Omega).$$

Since  $\partial \mathbb{P} \varphi \in L^1(\Omega)$  by Lemma A.1, the sequence  $\{v_{0,m}\}$  converges to a limit  $v_0$  in the distributional sense and the limit  $v_0$  satisfies  $(v_0, \varphi) = -(f, \partial \mathbb{P} \varphi)$ . Since the limit  $v_0$  is unique, the operator  $\mathbb{P}\partial$  is uniquely extendable for  $f \in W_0^{1,\infty}$ .

(ii) We recall that for a sequence  $\{v_{0,m}\}_{m=1}^\infty \subset L_\sigma^\infty$  satisfying

$$\begin{aligned} \|v_{0,m}\|_\infty &\leq K_1, \\ v_{0,m} &\rightarrow v_0 \quad \text{a.e. } \Omega, \end{aligned}$$

with some constant  $K_1$ , there exists a subsequence such that  $S(t)v_{0,m}$  converges to  $S(t)v_0$  locally uniformly in  $\overline{\Omega} \times (0, \infty)$  [4]. From the proof of Theorem 2.1, we observe that for a sequence  $\{f_m\} \subset W_0^{1,\infty}$  satisfying

$$\|f_m\|_{1,\infty} \leq K_2,$$

$$f_m \rightarrow f \quad \text{locally uniformly in } \overline{\Omega},$$

$\overline{S(t)\mathbb{P}\partial f_m}$  subsequently converges to  $\overline{S(t)\mathbb{P}\partial f}$  locally uniformly in  $\overline{\Omega} \times (0, \infty)$ .

(iii) The extension  $\overline{S(t)\mathbb{P}\partial}$  satisfies the property

$$S(t)\overline{S(s)\mathbb{P}\partial f} = \overline{S(t+s)\mathbb{P}\partial f}$$

for  $t \geq 0$ ,  $s > 0$  and  $f \in W_0^{1,\infty}$ . In fact, this property holds for  $f_m \in C_c^\infty$  satisfying (2.7). By choosing a subsequence,  $v_m(\cdot, t) = S(t)\mathbb{P}\partial f_m$  converges to  $v(\cdot, t) = S(t)\mathbb{P}\partial f$  locally uniformly in  $\overline{\Omega} \times (0, \infty)$  as in the proof of Theorem 2.1. For fixed  $s > 0$ , sending  $m \rightarrow \infty$  implies

$$\begin{aligned} S(t)S(s)\mathbb{P}\partial f_m &= S(t)v_m(s) \rightarrow S(t)v(s) \\ &= S(t)\overline{S(s)\mathbb{P}\partial f} \quad \text{locally uniformly in } \overline{\Omega} \times (0, \infty). \end{aligned}$$

Thus the property is inherited to  $\overline{S(t)\mathbb{P}\partial f}$ .

### 3. MILD SOLUTIONS ON $L^\infty_\sigma$

We prove Theorem 1.1 by approximation. We show that a sequence of mild solutions  $\{u_m\}$  subsequently converges to a limit  $u$  locally uniformly in  $\overline{\Omega} \times (0, T]$  by the  $L^\infty$ -estimates (1.5) and (1.6). Then, by an approximation argument for linear operators, we show that the limit  $u$  satisfies the integral equation (1.3). We first recall the existence of mild solutions on  $C_{0,\sigma}$  [1, Theorem 1.1]

**Proposition 3.1.** *For  $u_0 \in C_{0,\sigma}$ , there exist  $T \geq \varepsilon_0/\|u_0\|_\infty^2$  and a unique mild solution  $u \in C([0, T]; C_{0,\sigma})$  satisfying (1.3)-(1.6).*

We approximate  $u_0 \in L^\infty_\sigma$  by elements of  $C^\infty_{c,\sigma} \subset C_{0,\sigma}$ . We take a sequence  $\{u_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$  satisfying

$$(3.1) \quad \begin{aligned} \|u_{0,m}\|_\infty &\leq C\|u_0\|_\infty \\ u_{0,m} &\rightarrow u_0 \quad \text{a.e. in } \Omega, \end{aligned}$$

with some constant  $C$ , independent of  $m \geq 1$  [4, Lemma 5.1]. We apply Proposition 3.1 and observe that there exists  $T_m \geq \varepsilon_0/\|u_{0,m}\|_\infty^2$  and a unique mild solution  $u_m \in C([0, T_m]; C_{0,\sigma})$  satisfying

$$(3.2) \quad \begin{aligned} u_m(t) &= S(t)u_{0,m} - \int_0^t \overline{S(t-s)} \mathbb{P} \operatorname{div} F_m(s) ds, \\ F_m &= u_m u_m. \end{aligned}$$

Since  $T_m$  is estimated from below by (3.1), we take  $T \geq \varepsilon/\|u_0\|_\infty^2$  for  $\varepsilon = \varepsilon_0 C^{-2}/2$  so that  $T_m \geq T$  and  $u_m \in C([0, T]; C_{0,\sigma})$  for  $m \geq 1$ .

**Proposition 3.2.** *There exists a subsequence such that  $u_m$  converges to a limit  $u$  locally uniformly in  $\overline{\Omega} \times (0, T]$  together with  $\nabla u_m$ .*

*Proof.* It follows from (1.5), (1.6) and (3.1) that

$$(3.3) \quad \sup_{0 \leq t \leq T} \left\{ \|u_m\|_\infty(t) + t^{\frac{1}{2}} \|\nabla u_m\|_\infty(t) + t^{\frac{1+\beta}{2}} [\nabla u_m]_\Omega^{(\beta)}(t) \right\} \leq C'_1 \|u_0\|_\infty,$$

$$(3.4) \quad \sup_{x \in \Omega} \left\{ [u_m]_{[\delta, T]}^{(\gamma)}(x) + [\nabla u_m]_{[\delta, T]}^{(\frac{\gamma}{2})}(x) \right\} \leq C'_2 \|u_0\|_\infty,$$

for  $\beta, \gamma \in (0, 1)$  and  $\delta \in (0, T]$  with some constants  $C'_1$  and  $C'_2$ , independent of  $m \geq 1$ . Since  $u_m$  and  $\nabla u_m$  are uniformly bounded and equi-continuous in  $\overline{\Omega} \times [\delta, T]$ , the assertion follows from the Ascoli-Arzelà theorem.  $\square$

**Proposition 3.3.** *The limit  $u \in C_w([0, T]; L^\infty)$  is a mild solution for  $u_0 \in L^\infty_\sigma$ .*

*Proof.* We observe that the limit  $u$  satisfies (1.4) by sending  $m \rightarrow \infty$ . The estimates (3.3) and (3.4) are inherited to  $u$ . We prove that  $u$  satisfies the integral equation (1.3). By (3.1)

and choosing a subsequence,  $S(t)u_{0,m}$  converges to  $S(t)u_0$  locally uniformly in  $\overline{\Omega} \times (0, T]$  by Remarks 2.9 (ii). It follows from (3.3) and Proposition 3.2 that

$$(3.5) \quad \begin{aligned} \|F_m\|_\infty &\leq K, \\ \|\nabla F_m\|_\infty &\leq \frac{2}{s^{\frac{1}{2}}}K, \end{aligned}$$

$$F_m \rightarrow F \quad \text{locally uniformly in } \overline{\Omega} \times (0, T] \text{ as } m \rightarrow \infty,$$

for  $F = uu$  and  $K = C'_1\|u_0\|_\infty$ . By choosing a subsequence, we have

$$\overline{S(\eta)\mathbb{P}\operatorname{div}F_m} \rightarrow \overline{S(\eta)\mathbb{P}\operatorname{div}F} \quad \text{locally uniformly in } \overline{\Omega} \times (0, T],$$

for each  $s \in (0, t)$  as in Remarks 2.9 (ii). It follows from (3.5) and (2.5) that

$$\|\overline{S(t-s)\mathbb{P}\operatorname{div}F_m}\|_\infty \leq \frac{C}{(t-s)^{\frac{1-\alpha}{2}}} \left(1 + \frac{2}{s^{\frac{\alpha}{2}}}\right) K^2$$

for  $0 < s < t$  and  $\alpha \in (0, 1)$ . By the dominated convergence theorem, we have

$$\int_0^t \overline{S(t-s)\mathbb{P}\operatorname{div}F_m} ds \rightarrow \int_0^t \overline{S(t-s)\mathbb{P}\operatorname{div}F} ds \quad \text{locally uniformly in } \overline{\Omega} \times [0, T].$$

Thus sending  $m \rightarrow \infty$  implies that the limit  $u$  is a mild solution for  $u_0 \in L^\infty_\sigma$ . Since  $S(t)u_0$  is weakly-star continuous on  $L^\infty$  at  $t = 0$  [4], so is  $u$ .  $\square$

It remains to show continuity at  $t = 0$  for  $u_0 \in BUC_\sigma$ .

**Proposition 3.4.** *For  $u_0 \in BUC_\sigma$ ,  $S(t)u_0, t^{1/2}\nabla S(t)u_0 \in C([0, T]; BUC)$  and  $t^{1/2}\|\nabla S(t)u_0\|_\infty \rightarrow 0$  as  $t \rightarrow 0$ .*

*Proof.* Since  $S(t)$  is a  $C_0$ -analytic semigroup on  $BUC_\sigma$  [4],  $S(t)u_0 \in C([0, T]; BUC_\sigma)$ . Moreover,  $t^{1/2}\nabla S(t)u_0$  is continuous and bounded for  $t \in (0, T]$  in  $BUC$ . We show that  $t^{1/2}\|\nabla S(t)u_0\|_\infty \rightarrow 0$  as  $t \rightarrow 0$ .

We divide  $u_0$  into two terms by using the Bogovskii operator. For  $u_0 \in BUC_\sigma$ , there exists  $u_0^1 \in C_{0,\sigma}$  with compact support in  $\overline{\Omega}$  and  $u_0^2 \in BUC_\sigma$  supported away from  $\partial\Omega$  such that  $u_0 = u_0^1 + u_0^2$  (see [4, Lemma 5.1]). Let  $A$  denote the Stokes operator and  $D(A)$  denote the domain of  $A$  in  $BUC_\sigma$ . Since  $S(t)$  is a  $C_0$ -semigroup on  $BUC_\sigma$ ,  $D(A)$  is dense in  $BUC_\sigma$ . It follows from the resolvent estimate [5, Theorem 1.3] that

$$(3.5) \quad \|\nabla v\|_\infty \leq C(\|v\|_\infty + \|Av\|_\infty) \quad \text{for } v \in D(A).$$

We take an arbitrary  $\epsilon > 0$ . For  $u_0^1 \in C_{0,\sigma}$ , there exists  $\{u_{0,m}^1\} \subset C_{c,\sigma}^\infty$  such that  $\|u_0^1 - u_{0,m}^1\|_\infty \leq \epsilon$  for  $m \geq N_\epsilon^1$ . We apply (3.5) and observe that

$$\begin{aligned} t^{\frac{1}{2}}\|\nabla S(t)u_{0,m}^1\|_\infty &\leq t^{\frac{1}{2}}C(\|S(t)u_{0,m}^1\|_\infty + \|S(t)Au_{0,m}^1\|_\infty) \\ &\leq t^{\frac{1}{2}}C'(\|u_{0,m}^1\|_\infty + \|Au_{0,m}^1\|_\infty) \rightarrow 0 \quad \text{as } t \rightarrow 0. \end{aligned}$$

We estimate

$$\begin{aligned} \overline{\lim}_{t \rightarrow 0} t^{\frac{1}{2}}\|\nabla S(t)u_0^1\|_\infty &\leq \overline{\lim}_{t \rightarrow 0} t^{\frac{1}{2}}\|\nabla S(t)(u_0^1 - u_{0,m}^1)\|_\infty + t^{\frac{1}{2}}\|\nabla S(t)u_{0,m}^1\|_\infty \\ &\leq C''\epsilon. \end{aligned}$$

We set  $u_{0,m}^2 = \eta_{\delta_m} * u_0^2$  by the mollifier  $\eta_{\delta_m}$  so that  $u_{0,m}^2$  is smooth in  $\overline{\Omega}$  and  $\|u_0^2 - u_{0,m}^2\|_\infty \leq \epsilon$  for  $m \geq N_\epsilon^2$ . Since  $u_{0,m}^2$  is supported away from  $\partial\Omega$ , we have  $AS(t)u_{0,m}^2 = S(t)\Delta u_{0,m}^2$  (see [4, Proposition 6.1]). By a similar way as for  $u_0^1$ , we estimate  $\overline{\lim}_{t \rightarrow 0} t^{1/2} \|\nabla S(t)u_0^2\|_\infty \leq C''\epsilon$ . We proved

$$\overline{\lim}_{t \rightarrow 0} t^{\frac{1}{2}} \|\nabla S(t)u_0\|_\infty \leq 2C''\epsilon.$$

Since  $\epsilon > 0$  is arbitrary, we proved  $t^{1/2} \|\nabla S(t)u_0\|_\infty \rightarrow 0$  as  $t \rightarrow 0$ .  $\square$

*Proof of Theorem 1.1.* The assertion follows from Propositions 3.1-3.4. The proof is now complete.  $\square$

**Remark 3.5.** We set the associated pressure of mild solutions on  $L^\infty$  by (1.7) and the harmonic-pressure operator  $\mathbb{K} : L_{\text{tan}}^\infty(\partial\Omega) \rightarrow L_d^\infty(\Omega)$ , which is a solution operator of the homogeneous Neumann problem,

$$\begin{aligned} \Delta q &= 0 \quad \text{in } \Omega, \\ \frac{\partial q}{\partial n} &= \text{div}_{\partial\Omega} W \quad \text{on } \partial\Omega. \end{aligned}$$

Note that  $\Delta u \cdot n = \text{div}_{\partial\Omega} W$  by the divergence-free condition of  $u$ . Here,  $L_{\text{tan}}^\infty(\partial\Omega)$  denotes the space of all bounded tangential vector fields on  $\partial\Omega$  and  $L_d^\infty(\Omega)$  is the space of all functions  $f \in L_{\text{loc}}^1(\Omega)$  such that  $df$  is bounded in  $\Omega$  for  $d(x) = \inf_{y \in \partial\Omega} |x - y|$ ,  $x \in \Omega$ . Since  $W = -(\nabla u - \nabla^T u)n$  is bounded on  $\partial\Omega$  for mild solutions on  $L^\infty$ ,  $\nabla q = \mathbb{K}W$  is defined as an element of  $L_d^\infty$ . Moreover,  $\mathbb{Q}\text{div}F$  is uniquely defined for  $F = uu \in W_0^{1,\infty}$  as a distribution by Remarks 2.9 (i). Thus the associated pressure is defined by (1.7) for mild solutions on  $L^\infty$ .

#### ACKNOWLEDGEMENTS

The author is grateful to the anonymous referees for their valuable comments. This work was partially supported by JSPS through the Grant-in-aid for Research Activity Start-up 15H06312 and Kyoto University Research Funds for Young Scientists (Start-up) FY2015.

#### APPENDIX A. $L^1$ -ESTIMATES FOR THE NEUMANN PROBLEM

In Appendix A, we prove that  $\nabla \mathbb{P}\varphi \in L^1(\Omega)$ ,  $\varphi \in C_c^\infty(\Omega)$ , for an exterior domain  $\Omega$ . We first estimate  $L^1$ -norms of solutions for the Poisson equation in  $\mathbb{R}^n$  by using the heat semi-group. Then, we reduce the problem to the homogeneous Neumann problem and estimate solutions by a layer potential.

**Lemma A.1.** *Let  $\Omega$  be an exterior domain with  $C^2$ -boundary in  $\mathbb{R}^n$ ,  $n \geq 2$ . Then,  $\nabla \mathbb{P}\varphi \in L^1(\Omega)$  for  $\varphi \in C_c^\infty(\Omega)$ .*

We set  $\nabla\Phi = \mathbb{Q}\varphi$  for  $\mathbb{Q} = I - \mathbb{P}$ . It suffices to show that  $\nabla^2\Phi$  is integrable in  $\Omega$ . We recall that the  $\Phi$  solves the Neumann problem

$$(A.1) \quad \begin{aligned} \Delta\Phi &= \operatorname{div}\varphi \quad \text{in } \Omega, \\ \frac{\partial\Phi}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

See [14, Lemma III.1.2]. We observe that  $\Phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$  by the elliptic regularity theory (e.g., [26, Teor. 4.1]) since  $\varphi$  is smooth in  $\Omega$  and the boundary is  $C^2$ . We may assume that  $0 \in \Omega^c$  by translation. We take  $R > 0$  such that  $\Omega^c \subset B_0(R)$ . Let  $E$  denote the fundamental solution of the Laplace equation, i.e.,  $E(x) = C_n|x|^{-(n-2)}$  for  $n \geq 3$  and  $E(x) = -(2\pi)^{-1} \log|x|$  for  $n = 2$ , where  $C_n = (an(n-2))^{-1}$  and  $a$  denotes the volume of  $n$ -dimensional unit ball. We first show that the statement of Lemma A.1 is valid for  $\Omega = \mathbb{R}^n$ . In the sequel, we do not distinguish  $\varphi \in C_c^\infty(\Omega)$  and its zero extension to  $\mathbb{R}^n \setminus \Omega$ .

**Proposition A.2.** *Set  $h = E * \varphi$  and  $\Phi_1 = -\operatorname{div} h$ . Then,  $\nabla^3 h$  is integrable in  $\mathbb{R}^n$ . In particular,  $\nabla^2\Phi_1 \in L^1(\mathbb{R}^n)$ .*

*Proof.* By using the heat semigroup, we transform  $h$  into

$$h = \int_0^\infty e^{t\Delta} \varphi dt.$$

We divide  $h$  into two terms and observe that

$$\partial_x^3 h = \int_0^1 \partial_x e^{t\Delta} \partial_x^2 \varphi dt + \int_1^\infty \partial_x^3 e^{t\Delta} \varphi dt,$$

where  $\partial_x = \partial_{x_j}$  indiscriminately denotes the spatial derivatives for  $j = 1, \dots, n$ . We estimate

$$\begin{aligned} \|\partial_x^3 h\|_{L^1(\mathbb{R}^n)} &\lesssim \int_0^1 \frac{1}{t^{1/2}} \|\partial_x^2 \varphi\|_{L^1(\mathbb{R}^n)} dt + \int_1^\infty \frac{1}{t^{3/2}} \|\varphi\|_{L^1(\mathbb{R}^n)} dt \\ &\lesssim \|\partial_x^2 \varphi\|_{L^1(\mathbb{R}^n)} + \|\varphi\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

We proved  $\nabla^3 h \in L^1(\mathbb{R}^n)$ . □

We reduce (A.1) to the homogeneous Neumann problem

$$(A.2) \quad \begin{aligned} -\Delta\Phi_2 &= 0 \quad \text{in } \Omega, \\ \frac{\partial\Phi_2}{\partial n} &= g \quad \text{on } \partial\Omega. \end{aligned}$$

We write connected components of  $\Omega$  by unbounded  $\Omega_0$  and bounded  $\Omega_1, \dots, \Omega_N$ , i.e.,  $\Omega = \Omega_0 \cup (\cup_{j=1}^N \Omega_j)$ .

**Proposition A.3.** *Set  $\Phi_2 = \Phi - \Phi_1$ . Then,  $\Phi_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$  solves (A.2) for  $g = \operatorname{div}_{\partial\Omega}(An)$  and  $A = \nabla h - \nabla^T h$ . The function  $g \in C(\partial\Omega)$  satisfies*

$$(A.3) \quad \int_{\partial\Omega_j} g d\mathcal{H} = 0 \quad \text{for } j = 0, 1, \dots, N.$$

*Proof.* We observe that  $\Phi_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$  satisfies  $-\Delta\Phi_2 = 0$  in  $\Omega$  and  $\partial\Phi_2/\partial n = \partial(\operatorname{div} h)/\partial n$  on  $\partial\Omega$ . We take an arbitrary  $\rho \in C_c^\infty(\mathbb{R}^n)$ . Since  $An = (\sum_{1 \leq j \leq n} (\partial_j h^i - \partial_i h^j) n^j)_{1 \leq i \leq n}$  is a tangential vector field on  $\partial\Omega$  (i.e.,  $An \cdot n = 0$  on  $\partial\Omega$ ), applying integration by parts yields

$$\begin{aligned} \int_{\partial\Omega} g \rho d\mathcal{H} &= \int_{\partial\Omega} \operatorname{div}_{\partial\Omega}(An) \rho d\mathcal{H} \\ &= - \int_{\partial\Omega} (An) \cdot \nabla \rho d\mathcal{H} \\ &= - \int_{\partial\Omega} (\partial_j h^i - \partial_i h^j) n^j \partial_i \rho d\mathcal{H} \\ &= - \int_{\partial\Omega} \partial_j h^i n^i \partial_i \rho d\mathcal{H} + \int_{\partial\Omega} \partial_j h^i n^i \partial_j \rho d\mathcal{H}, \end{aligned}$$

where the symbol of summation is suppressed. By integration by parts, we have

$$\begin{aligned} \int_{\partial\Omega} \partial_j h^i n^i \partial_i \rho d\mathcal{H} &= \int_{\partial\Omega} (\Delta h^i \partial_i \rho + \nabla h^i \cdot \nabla \partial_i \rho) dx \\ &= \int_{\partial\Omega} (\Delta h^i \partial_i \rho - \nabla \operatorname{div} h \cdot \nabla \rho) dx + \int_{\partial\Omega} \nabla h^i \cdot \nabla \rho n^i d\mathcal{H}. \end{aligned}$$

Since  $-\Delta h = \varphi$  is supported in  $\Omega$ , it follows that

$$\begin{aligned} \int_{\partial\Omega} g \rho d\mathcal{H} &= - \int_{\Omega} (\Delta h - \nabla \operatorname{div} h) \cdot \nabla \rho dx \\ &= - \int_{\Omega} (\Delta h \cdot \nabla \rho + \Delta \operatorname{div} h \rho) dx + \int_{\partial\Omega} \frac{\partial}{\partial n} \operatorname{div} h \rho d\mathcal{H} \\ &= \int_{\partial\Omega} \frac{\partial}{\partial n} \operatorname{div} h \rho d\mathcal{H}. \end{aligned}$$

Since  $\partial\Omega$  is  $C^2$ ,  $n$  is extendable to a  $C^1$ -function in a tubular neighborhood of  $\partial\Omega$ . Thus,  $g$  is continuous on  $\partial\Omega$ . Since  $\rho \in C_c^\infty(\mathbb{R}^n)$  is arbitrary, we proved  $\partial(\operatorname{div} h)/\partial n = g$  on  $\partial\Omega$ . Since  $g$  is a surface-divergence form, by integration by parts, (A.3) follows. The proof is complete.  $\square$

We estimate  $\Phi_2$  by means of the layer potential.

**Proposition A.4.** (i) For  $g \in C(\partial\Omega)$  satisfying (A.3), there exists a moment  $h \in C(\partial\Omega)$  satisfying  $\int_{\partial\Omega} h d\mathcal{H} = 0$  and

$$-g(x) = \frac{1}{2}h(x) + \int_{\partial\Omega} n(x) \cdot \nabla_x E(x-y) h(y) d\mathcal{H}(y) \quad x \in \partial\Omega.$$

(ii) Set the single layer potential

$$\tilde{\Phi}_2(x) = - \int_{\partial\Omega} E(x-y) h(y) d\mathcal{H}(y).$$

Then,  $\tilde{\Phi}_2$  is continuous in  $\overline{\Omega}$ . Moreover, the normal derivative  $\partial_n \tilde{\Phi}_2$  exists and is continuous on  $\partial\Omega$ . The function  $\tilde{\Phi}_2$  satisfies (A.2) and decays as  $|x| \rightarrow \infty$ .

*Proof.* The assertion (i) is based on the Fredholm's theorem. See [12, (3.40), (3.13), (3.30)]. Since  $h$  is bounded on  $\partial\Omega$ ,  $\tilde{\Phi}_2$  is continuous in  $\overline{\Omega}$ . Moreover, we have

$$-\frac{\partial \tilde{\Phi}_2}{\partial n}(x) = \frac{1}{2}h(x) + \int_{\partial\Omega} n(x) \cdot \nabla E(x-y)h(y)d\mathcal{H}(y) \quad x \in \partial\Omega.$$

See [12, (3.25), (3.28)]. Thus  $\tilde{\Phi}_2$  satisfies (A.2) by the assertion (i). When  $n \geq 3$ ,  $\tilde{\Phi}_2(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  since the fundamental solution decays as  $|x| \rightarrow \infty$ . Moreover, when  $n = 2$ , the average of  $h$  on  $\partial\Omega$  is zero and we have

$$\begin{aligned} \tilde{\Phi}_2(x) &= - \int_{\partial\Omega} (E(x-y) - E(x))h(y)d\mathcal{H}(y) \\ &= \frac{1}{2\pi} \int_{\partial\Omega} \log\left(\frac{|x-y|}{|x|}\right)h(y)d\mathcal{H}(y) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

The proof is complete.  $\square$

**Proposition A.5.** *The function  $\tilde{\Phi}_2$  agrees with  $\Phi_2$  up to constant.*

*Proof.* Since  $\nabla\Phi_2 = \nabla\Phi - \nabla\Phi_1$  is  $L^p$ -integrable in  $\Omega$  for all  $p \in (1, \infty)$  (e.g., [31]), we may assume that  $\Phi_2 \rightarrow 0$  as  $|x| \rightarrow \infty$  by shifting  $\Phi_2$  by a constant. We set  $\Psi = \Phi_2 - \tilde{\Phi}_2$  and observe that  $\Psi$  is continuous in  $\overline{\Omega}$ . Moreover, the normal derivative exists and is continuous on  $\partial\Omega$  by Proposition A.4. The function  $\Psi$  satisfies  $-\Delta\Psi = 0$  in  $\Omega$ ,  $\partial\Psi/\partial n = 0$  on  $\partial\Omega$  and  $\Psi \rightarrow 0$  as  $|x| \rightarrow \infty$ . By the elliptic regularity theory [26],  $\Psi$  is smooth in  $\Omega$  and continuously differentiable in  $\overline{\Omega}$ .

We shall show that  $\Psi \equiv 0$ . Since  $\Psi$  decays as  $|x| \rightarrow \infty$ , there exists a point  $x_0 \in \overline{\Omega}$  such that  $\sup_{x \in \Omega} \Psi(x) = \Psi(x_0)$ . Suppose that  $x_0 \in \partial\Omega$ . Since the boundary of class  $C^2$  satisfies the interior sphere condition, the Hopf's lemma [30, Chapter 2 Theorem 7] implies that  $\partial\Psi(x_0)/\partial n > 0$ . Thus  $x_0 \in \Omega$ . We apply the strong maximum principle [30, Chapter 2 Theorem 5] and conclude that  $\Psi$  is constant. Since  $\Psi$  decays as  $|x| \rightarrow \infty$ , we have  $\Psi \equiv 0$ . The proof is complete.  $\square$

**Proposition A.6.**  *$\nabla^2\Phi_2$  is integrable in  $\Omega$ .*

*Proof.* Since  $\nabla^2\Phi_2$  is integrable near the boundary  $\partial\Omega$ , it suffices to show that  $\nabla^2\Phi_2 \in L^1(\{|x| \geq 2R\})$ . Since  $h \in C(\partial\Omega)$  satisfies  $\int_{\partial\Omega} h d\mathcal{H} = 0$ , we observe that

$$\begin{aligned} \tilde{\Phi}_2(x) &= - \int_{\partial\Omega} (E(x-y) - E(x))h(y)d\mathcal{H}(y) \\ &= \int_0^1 dt \int_{\partial\Omega} y \cdot (\nabla E)(x-ty)h(y)d\mathcal{H}(y). \end{aligned}$$

Since  $\Omega^c \subset B_0(R)$ , for  $|x| \geq 2R$  we observe that

$$\begin{aligned} |x-ty| &\geq \left| |x| - |ty| \right| \\ &\geq |x| - R \\ &\geq \frac{|x|}{2}. \end{aligned}$$

Since  $\tilde{\Phi}_2$  agrees with  $\Phi_2$  up to constant, we estimate

$$\begin{aligned} |\nabla^2 \Phi_2(x)| &\lesssim \int_0^1 dt \int_{\partial\Omega} \frac{|h(y)|}{|x - ty|^{n+1}} d\mathcal{H}(y) \\ &\lesssim \frac{1}{|x|^{n+1}} \|h\|_{L^1(\partial\Omega)}. \end{aligned}$$

Thus,  $\nabla^2 \Phi_2$  is integrable in  $\{|x| \geq 2R\}$ . The proof is complete.  $\square$

*Proof of Lemma A.1.* By Propositions A.2 and A.6, the assertion follows.  $\square$

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